Mankiw blog problem,
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"Imagine that you start off with a portfolio of 60 percent stocks and 40 percent bonds. The returns on stocks, bonds, and gold are uncorrelated. Stocks earn a higher expected return than bonds. Bonds and gold earn the same lower expected return, but gold returns are three times as volatile as bond returns, as measured by the standard deviation. You want to minimize risk, measured by the variance of your portfolio return, without changing the expected return on your portfolio. How much gold should you buy?"

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Initial portfolio expected return:
$\mu_p^{\text{initial}} = \mathbb{E}[R_p^{\text{initial}}] = \mathbb{E}[.6R_S + .4R_B] = .6\mu_S + .4\mu_B$
Assume $\mu_S > \mu_B = \mu_G$, and $\sigma_G = 3\sigma_B$

$$\begin{aligned}
\text{Var}[R_p] &= \text{Var}[\alpha_S R_S + \alpha_B R_B + \alpha_G R_G] \\
&= \alpha_S^2\sigma_S^2 + \alpha_B^2\sigma_B^2 + \alpha_G^2\sigma_G^2 && \text{uncorrelated} \\
&= \alpha_S^2\sigma_S^2 + \alpha_B^2\sigma_B^2 + \alpha_G^2(3\sigma_B)^2 \\
&= \alpha_S^2\sigma_S^2 + \alpha_B^2\sigma_B^2 + \alpha_G^2 9\sigma_B^2
\end{aligned}$$

Goal:
$\min_{\alpha_S,\alpha_B,\alpha_G} \text{Var}[R_p]$
s.t. $\mathbb{E}[R_p] = .6\mu_S + .4\mu_B$ and $\alpha_S + \alpha_B + \alpha_G = 1$

simplify the first constraint:

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\begin{aligned}
\mathbb{E}[R_p] &= \alpha_S\mu_S + \alpha_B\mu_B + \alpha_G\mu_G \\
&= \alpha_S\mu_S + \alpha_B\mu_B + \alpha_G\mu_B && \text{bonds and gold have same return} \\
&= \alpha_S\mu_S + (\alpha_B + \alpha_G)\mu_B \\
&= \alpha_S\mu_S + (1 - \alpha_S)\mu_B && \alpha_S + \alpha_B + \alpha_G = 1 \\
&= \alpha_S\mu_S + \mu_B - \alpha_S\mu_B \\
&= \alpha_S(\mu_S - \mu_B) + \mu_B
\end{aligned}
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To keep the expected return equal to the initial expected return:
$\alpha_S(\mu_S - \mu_B) + \mu_B = .6\mu_S + .4\mu_B$
$\Leftrightarrow \alpha_S(\mu_S - \mu_B) = .6\mu_S - .6\mu_B$
$\Leftrightarrow \alpha_S(\mu_S - \mu_B) = .6(\mu_S - \mu_B)$
$\Leftrightarrow \alpha_S = .6$
This agent should keep 60% of his portfolio in stocks.
Rewrite the agent's problem:
$\min_{\alpha_B,\alpha_G} (.6)^2\sigma_S^2 + \alpha_B^2\sigma_B^2 + (1 - .6 - \alpha_B)^2 9\sigma_B^2$
FOC: $2\alpha_B\sigma_B^2 + 2(1 - .6 - \alpha_B)(-1)9\sigma_B^2 = 0$
$\Leftrightarrow \alpha_B = (1 - .6 - \alpha_B)9$
$\Leftrightarrow \alpha_B + 9\alpha_B = (1 - .6)9$
$\Leftrightarrow 10\alpha_B = (.4)9$
$\Leftrightarrow \alpha_B = .36$
Finally we find the investment in gold:
$\alpha_G = 1 - .6 - \alpha_B = 1 - .6 - .36 = .04$

Initially the investor's portfolio is:
$\boxed{\alpha_S^{\text{initial}} = .6, \alpha_B^{\text{initial}} = .4}$
With gold the investor's new portfolio is:
$\boxed{\alpha_S = .6, \alpha_B = .36, \alpha_G = .04}$
Intuition: even though gold has the same expected return as bonds with 3 times as much risk, an investor will still choose to hold some gold for diversification purposes. Especially because it is uncorrelated with stocks and bonds.